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A DUAL FORMULATION OF CYCLIC SYMMETRY: APPLICATION IN FREE VIBRATION ANALYSIS

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Abstract. *In this work, the cyclic symmetry boundary condition between sectors of a rotationally periodic structure are reformulated and the compatibility condition is imposed by means of Lagrange multipliers. In the augmented system, the constraint matrix carries the dependence on the nodal diameters (ND) whereas stiffness and mass matrices do not vary with ND. The Dual assembly formulation is exploited in the computation of eigenvectors and eigenvalues using a Preconditioned Projected Arnoldi Algorithm. The complex constraint matrix is used to build a projection operator, which imposes the compatibility constraint between the left and right side of the periodic geometry. Because the complex constraint matrix has orthogonal rows, the projector operator is easily assembled without the need of matrix inversion. The explicit factorization of the projected stiffness matrix is not required since Preconditioned Conjugated Gradient algorithm is used to solve the linear system during the inverse Arnoldi iteration step. Consequently, all ND share the same preconditioner reducing the global cost of eigenpairs computation, especially when multiple ND solutions are required. Five different preconditioners are tested to evaluate the numerical scalability of the linear solution: Identity, Projected Incomplete Chosleky, Projected Cholesky, Incomplete Cholesky and Cholesky. The Preconditioned Projected Arnoldi Algorithm is implemented in Python using the Scipy library which provides a wrapper for the Implicity Restarted Arnoldi method implemented in ARPACK. The projection and preconditioner steps are carried-out by Scipy sparse linear operator. The method is applied in 2D and 3D finite element models. The preconditioners studied can dramatically reduce the number of iterations when compared to CG without preconditioning. The full Cholesky preconditioner is on average 8 times faster than the identity precondicioner in the 2D, whereas it is approximately 16 faster in the 3D case. The factorizing is performed only once, and its computational cost can be divided among the different NDs, which makes the method especially advantageous for geometries with a large number of sectors.*

1 INTRODUCTION

Free vibration analysis plays an important role in understanding the dynamics of structures. It provides the natural frequencies which must be identified in order to avoid catastrophic failures due to resonances. Moreover, eigenmodes can be used to reduce the number of degrees of freedom by using Reduced Order Models (ROM) such as Craig-Bampton [1], Dual-Craig-Bampton [2], Rubin [3], among others. Despite its advantages, full eigenanalysis is very costly, which means that in many cases only the modes related to the lowest frequencies are relevant for practical problems. Therefore, Lanczos [4], and implicit restarted Arnoldi algorithms [5] are often used as eigensolvers due to their efficiency. The major cost of the algorithms mentioned lies in the inverse iteration, where the solution of a linear system is required.

When structures such as turbines, propellers, and gears are analysed, the property of rotational periodic symmetry is often utilized, reducing the computational cost of the linear system as well as the memory required to store the system matrices. Figure 1 shows a cyclic structure with the highlighted reference sector used for analyzing the whole geometry. In such cases, a Discrete Fourier Transform is applied in the equation of motion and in the compatibility condition between neighbors, which reformulates the problem in terms of harmonic indices. The speed-up is achieved by means of a harmonic decoupling among sectors when imposing the periodic condition by eliminating the constrained degrees of freedom, namely, primal assembly. In the work developed in[6], the cyclic symmetry is imposed by complex constraints relating the left to the right side. Considering a periodic structure composed of N sectors, which has θ as the sector angle, the classical cyclic boundary condition creates approximately $N/2$ different systems of equation to be solved independently. The number of variables is not changed by the harmonic transformation, because the transformed system has complex degrees of freedom. Therefore, the cyclic symmetry should not be understood as a reduction technique, since it is only a transformation operation. However, the decoupling provided by the harmonic decomposition results in the ability of solving independent problems for a given set of harmonic indices, also known as Nodal Diameter (ND). If cyclic symmetry is considered for ROM, not only a subset of modes must be selected but also a subset of nodal diameters. This is usually done by analyzing the possible excitations, using the Campbell diagram or SAFE diagram, see [7].

In this paper, the primal and dual assembly of cyclic constraints are presented using a substructuring framework and applied to the computation of the eigen-pairs of a cyclic structure. In the dual formulation, the cyclic constraints are described with a complex constraint matrix thereby rendering the system matrices, namely stiffness, and mass, unaffected. Due to the hybrid nature of the eigen-value problem in the dual formulation (displacements and Lagrange multipliers), the conventional eigen-solvers are not suitable. Therefore, a modified version of Projected Arnoldi algorithm is employed to efficiently solve the dual cyclic formulation. The novel approach introduces a preconditioning step in the inverse iteration of the Arnoldi process, which uses the reference sector without constraints. Five different preconditioners are tested to evaluate the numerical scalability of the linear solution, namely: Identity, Projected Incomplete Chosleky, Projected Cholesky, Incomplete Cholesky and Cholesky.

In the remaining sections the classical cyclic symmetry is formally introduced and the dual formulation is presented. The projected Arnoldi method is described, eigen-analysis is performed using the complex cyclic constraints and its projection operator. Three different Finite element test-cases are presented in order to show the application of the new formulation.

2 Classical Cyclic Symmetry Formulation

In this section, the classical theory behind cyclic symmetry boundary condition is summarized. Let us consider a geometry composed of N identical sectors, as shown in the Figure 1. If each sector has its own local coordinate system and no mistuning is considered, the equation of motion together with the compatibility constraint can be written as:

$$\begin{aligned} M\ddot{u}^s + Ku^s &= f^s + g^s \qquad s = 1, ..., N \\ Tu_l^s &= u_r^{s+1} \\ Tg_l^s &= -g_r^{s+1} \end{aligned} \tag{1}$$

where M and K are the mass and stiffness matrices respectively, f is the external force vector, g is the connecting forces among neighbors, and superscript s represents the index of a sector. The second expression in (1) represents the compatibility condition among sectors, where T is a rotation matrix, and the subscripts l and r represent the interface dofs of the left and right respectively.

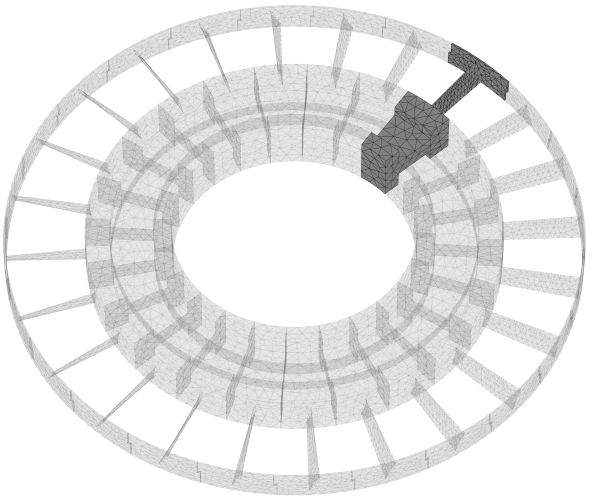

Figure 1: Cyclic Structure and the highlighted reference sector.

It is important to notice that given the choice of local coordinate system, the stiffness and mass matrices are the same for all sectors, however the transformation matrix T is required to define a proper compatibility condition. Applying a Discrete Fourier Transform in space $\{\mathcal{F} : u^s \rightarrow u^n \quad | \quad \sum_{s=1}^{N} u^s e^{-jn\theta s}\}$ in (1) and rewriting the displacement solution, interface force and external forces in terms of Fourier Series:

$$\begin{aligned} u^s &= \frac{1}{\bar{N}} \sum_{n=0}^{\bar{N}} u^n e^{jn\theta s} \\ g^s &= \frac{1}{\bar{N}} \sum_{n=0}^{\bar{N}} g^n e^{jn\theta s} \\ f^s &= \frac{1}{\bar{N}} \sum_{n=0}^{\bar{N}} f^n e^{jn\theta s} \end{aligned} \tag{2}$$

where $j = \sqrt{-1}$ is the imaginary unit, $\theta = \frac{2\pi}{N}$ is the sector angle, n is the nodal diameter index, and $\bar{N}$ is the largest nodal diameter index such that:

$$\begin{aligned} \bar{N} &= \frac{N}{2} - 1 \quad \text{if } N \text{ is even} \\ \bar{N} &= \frac{N-1}{2} \quad \text{if } N \text{ is odd} \end{aligned} \tag{3}$$

equation (1) becomes (4).

$$\begin{aligned} M\ddot{u}^n + Ku^n &= f^n + g^n \qquad n = 1, ..., \bar{N} \\ Tu_l^n &= u_r^{n+1} = u_r^n e^{jn\theta} \\ Tg_l^n &= -g_r^{n+1} = -g_r^n e^{jn\theta} \end{aligned} \tag{4}$$

If a permutation matrix is applied in the displacement degrees of freedom of a sector such that one defines a block vector $u = [u_r, u_l, u_i]$, where the indices r, l and i are the right, left and interior dofs respectively, equation (4) can be rewritten using block matrices:

$$\begin{bmatrix} M_{rr} & M_{rl} & M_{ri} \\ M_{lr} & M_{ll} & M_{li} \\ M_{ir} & M_{il} & M_{ii} \end{bmatrix} \begin{bmatrix} \ddot{u}_r^n \\ \ddot{u}_l^n \\ \ddot{u}_i^n \end{bmatrix} + \begin{bmatrix} K_{rr} & K_{rl} & K_{ri} \\ K_{lr} & K_{ll} & K_{li} \\ K_{ir} & K_{il} & K_{ii} \end{bmatrix} \begin{bmatrix} u_r^n \\ u_l^n \\ u_i^n \end{bmatrix} = \begin{bmatrix} f_r^n \\ f_l^n \\ f_i^n \end{bmatrix} + \begin{bmatrix} g_r^n \\ g_l^n \\ g_i^n \end{bmatrix} \tag{5}$$

Classically, the left degrees of freedom are removed by replacing the harmonic compatibility presented in equation (4) into equation (5):

$$\begin{bmatrix} M_{rr} + TM_{ll}T^T + e^{jn\theta}M_{rl}T^T + e^{-jn\theta}TM_{lr} & M_{ri} + e^{-jn\theta}TM_{li} \\ M_{ir} + e^{jn\theta}M_{il}T^T & M_{ii} \end{bmatrix} \begin{bmatrix} \ddot{u}_r^n \\ \ddot{u}_i^n \end{bmatrix} + \begin{bmatrix} K_{rr} + TK_{ll}T^T + e^{jn\theta}K_{rl}T^T + e^{-jn\theta}TK_{lr} & K_{ri} + e^{-jn\theta}TK_{li} \\ K_{ir} + e^{jn\theta}K_{il}T^T & K_{ii} \end{bmatrix} \begin{bmatrix} u_r^n \\ u_i^n \end{bmatrix} = \begin{bmatrix} f_r^n + e^{-jn\theta}Tf_l^n \\ f_i \end{bmatrix} \tag{6}$$

Equation (6) is the constrained equation of motion, which contains only the right and interior degrees of freedom. Using the property $TT^T = I$, the elimination process can be carried out by defining a complex matrix L_n which lies in the null space of the constraints.

$$L_n = \begin{bmatrix} I & 0 \\ e^{jn\theta}T^T & 0 \\ 0 & I \end{bmatrix} \tag{7}$$

Therefore, the primal assembly of cyclic symmetry can be written using a substructuring framework as presented in [8]:

$$M_n\ddot{u}_{\text{primal}}^n + K_n u_{\text{primal}}^n = L_n^H f^n \qquad n = 1, ..., \bar{N} \tag{8}$$

where $M_n \equiv (L_n^H M L_n)$ and $K_n \equiv (L_n^H K L_n)$, and the superscript H represents the adjoint matrix. It can be readily verified that (8) leads to the form (6).

For free vibration problems, one sets $f^n = 0$ and $u_{\text{primal}}^n = \phi_{\text{primal}}^n e^{\omega t}$, which leads to the generalized eigenvalue problem:

$$[K_n - \omega^2 M_n]\phi_{\text{primal}}^n = 0 \qquad n = 1, ..., \bar{N} \tag{9}$$

where the subscript n represents the dependency on the nodal diameter, ϕ^n_{primal} is a complex eigenvector, and K_n and M_n are Hermitian matrices. If the full set of eigenvector is required, all nodal diameter indices in (9) must be evaluated.

3 Dual Cyclic Symmetry Formulation

The classical treatment of cyclic symmetry formulation modifies the system operator, namely stiffness and mass matrices, by means of the primal assembly operator L_n as shown above. This results in a set of unique variables defined as u_{primal}. In this section, the dual formulation of (4) is presented. First, let us define two Boolean operators to extract 'left' and 'right' dofs as classical FETI methods [9].

$$\begin{aligned} u_l &= B_l u \\ u_r &= B_r u \end{aligned} \tag{10}$$

For the re-arranged displacement vector $u = [u_r, u_l, u_i]$ the Boolean operators are defined by $B_r = [I\ 0\ 0]$ and $B_l = [0\ I\ 0]$. Rewritting the compatibility constraint in (4) by using the Boolean definition in (10):

$$C_n u = [B_r - e^{-jn\theta} T B_l] u = 0 \tag{11}$$

where C_n is a linear complex cyclic constraint operator. It is easy to show that $L_n^H C_n^H = 0$. Therefore, a Lagrange multiplier must be introduced in the equation of motion in order to satisfy the compatibility constraint, such that:

$$\begin{aligned} M\ddot{u}^n + Ku^n &= f^n - C_n^H \lambda^n \qquad n = 1, ..., \bar{N} \\ C_n u^n &= 0 \end{aligned} \tag{12}$$

Or in matrix notation:

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{u}^n \\ \lambda^n \end{bmatrix} + \begin{bmatrix} K & C_n^H \\ C_n & 0 \end{bmatrix} \begin{bmatrix} u^n \\ \lambda^n \end{bmatrix} = \begin{bmatrix} f^n \\ 0 \end{bmatrix} \qquad n = 1, ..., \bar{N} \tag{13}$$

where C_n^H is the adjoint matrix of the cyclic constraint operator.

The above expression is the dual assembly formulation for cyclic symmetry boundary condition. This equation contains not only displacement dofs but also Lagrange multipliers to enforced continuity among sectors. Clearly, this results in a system which contains more unknowns than the formulation in (8), however the stiffness and mass matrices do not depend on the nodal diameter n. Therefore, this property can be exploited using dual Schur complement solvers such as described in [10]. A similar technique for cyclic problems was already applied in the work developed in [11]. However, in [11] the Lagrange multipliers are connected to sectors as substructures, whereas in this work, λ^n is defined in the harmonic space. The novel approach can be seen as a decoupling of the interface displacement and forces generated among neighboring sectors. Besides that, the formulation gives the explicit harmonic gap, namely $C_n u^n$, which can be useful when the interface connection between sector is nonlinear.

3.1 Projected Arnoldi for Cyclic Symmetry Modal Analysis

The free vibration analysis of the system (13) results in the following hybrid generalized eigenvalue problem:

$$\begin{bmatrix} K & C_n^H \\ C_n & 0 \end{bmatrix} \begin{bmatrix} \phi'_u \\ \phi_\lambda \end{bmatrix} = \omega^2 \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \phi'_{u_1} \\ \phi_\lambda \end{bmatrix} \tag{14}$$

The dimension of the above problem is n_c+n_{dofs}, where n_c is the number of cyclic symmetry constraints, and n_{dofs} is the total number of dofs in the reference sector. The eigenpair solutions $(\omega_i, [\phi'_{u_i}, \phi_{\lambda_i}])$ have the form:

$$\left(\omega_1^u, \begin{bmatrix} \phi'_{u_1} \\ \phi_{\lambda_1} \end{bmatrix}\right), ..., \left(\omega_{n_{dofs}}^u, \begin{bmatrix} \phi'_{u_{n_{dofs}}} \\ \phi_{\lambda_{n_{dofs}}} \end{bmatrix}\right), \left(+\infty, \begin{bmatrix} 0 \\ \phi_{\lambda_{(n_{dofs}+1)}} \end{bmatrix}\right), ..., \left(+\infty, \begin{bmatrix} 0 \\ \phi_{\lambda_{(n_{dofs}+nc)}} \end{bmatrix}\right) \tag{15}$$

Note that, the eigenvector ϕ'_{u_i} lies in the null space of the complex cyclic symmetry constraint $C_n\phi'_{u_i} = 0$. In [12] some techniques are presented to solve the constrained eigenvalue problem presented in (14) , but it does not explore the structure of the problem to accelerate the inverse iteration of the eigen-solver. In order to solve the eigenvalue problem efficiently and profit from the independence of stiffness and mass matrices of the nodal diameter n, a constrained Krylov subspace is built based on the complex cyclic constraint operator, see [13]. The Krylov matrix for the generalized eigenvalue problem without constraints is defined as:

$$V_m = [b,\ Db,\ D^2b,\ ...\ ,\ D^{m-1}b] \tag{16}$$

where $D \equiv (K-\sigma M)^{-1}M$, which is the inverse shift operator such that $D : \mathbb{R}^n \to \mathbb{R}^n$, $b \in \mathbb{R}^n$. When a constraint $h(b) = 0$ such that $h : \mathbb{R}^n \to \mathbb{R}^{n_c}$ is defined, a Krylov basis vector b' must lie in the set of admissible vectors, mathematically $\Omega = \{b' \in \mathbb{R}^n \mid h(b') = 0\}$ as represented in Figure 2. Thus, the constrained Krylov matrix has the form:

$$V_m^c = [b',\ (Db')',\ (D^2b')',\ ...\ ,\ (D^{m-1}b')'] \tag{17}$$

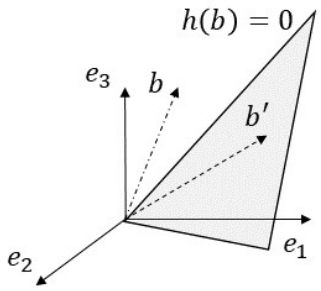

Figure 2: Vectors b and b' to build constrained Krylov subspace.

In the case of cyclic symmetry, one has a linear constraint, defined by $C_nu = 0$. One can write a projection operation P_n into the $Null(C_n)$ such that $P_n : b \to b'$, where the constraint $C_nb' = 0$ always holds. The projection matrix is explicitly written as:

$$P_n = I - C_n^H(C_nC_n^H)^{-1}C_n \tag{18}$$

where the subscript n refers to the nodal diameter dependency in the constraint matrix C_n. It is easy to show that $(C_n C_n^H)^{-1} = \frac{1}{2} I$, therefore the projection matrix can be written without matrix inversion as shown below:

$$P_n = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} - \frac{1}{2} \begin{bmatrix} I & -e^{-jn\theta} T & 0 \\ -e^{jn\theta} T^T & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{19}$$

The generalized constrained eigenvalue problem (14) can be rewritten as:

$$K\phi'_u + C_n^H \phi_\lambda = \omega^2 M \phi'_u \tag{20}$$

where ϕ'_u is a vector which respects the cyclic symmetry constraint. If one projects equation (20) in the feasible space using P_n^H and uses the relation $\phi'_u = P_n \phi_u$ one finally derives the projected eigenvalue problem:

$$P_n^H K P_n \phi_u = \omega^2 P_n^H M P_n \phi_u \tag{21}$$

where $\phi_u \in R^n$. Therefore a projected Arnoldi algorithm can be used to build the constrained orthonormal Krylov bases vectors. The projected Arnoldi method is exploited in [14] for updating eigenvalues in nonlinear dynamic problems. The procedure is summarized in Algorithm 1.

Algorithm 1 Projected Arnoldi Iteration

• **Parameters:** K, M, P_n and n_{int}.

1: **Initialization**
Compute $M_n^P = P_n^H M P_n$ and $K_n^P = P_n^H K P_n$
Start with an arbitrary vector ϕ_u^0 and $\beta_0 = 0$

2: **First step iteration**
Compute a projected unitary vector $v_n^1 = \frac{M_n^P \phi_u^0}{||\phi_u^0||_{M_n^P}}$
Solve the static-like problem $K_n^P \hat{\phi}_u^1 = v_n^1$
Compute $\alpha_1 = \hat{\phi}_u^{1T} v_n^1$
Compute a new orthogonal constrained Krylov vector ${\phi_u}^1 = \hat{\phi}_u^1 - \alpha^1 v_n^1$

3: **For** $j = 2, ..., n_{int}$
compute $\beta_j = ||\phi_u^{(j-1)}||_{M_n^p}$
if $\beta_j \neq 0$ then compute $v_n^j = M_n^P \phi_u^{(j-1)} / \beta_j$
Solve the static like problem $K_n^P \hat{\phi}_u^j = v_n^j$
Compute $\alpha_j = \hat{\phi}_u^{jT} v_n^j$
Compute a new constrained Krylov vector $\phi^{j+1} = \hat{\phi}_u^j - \alpha^j v_n^j - \beta_j v_n^{(j-1)}$

The bottleneck of Algorithm (1) is the solution of the singular linear system $K_n^P \hat{\phi}_u^j = v_n^j$. In this work, the linear system is solved by a Preconditioned Conjugate Gradient (PCG) algorithm. The preconditioner $\bar{K}_n^P$ is an estimation of the generalized pseudoinverse of $(K_n^P)^{-1}$. A set of preconditioners are constructed such that, they can be reutilized for all nodal diameters. In this paper, 5 different preconditioners are studied:

- Identity : I
- Incomplete Cholesky : $\hat{K}^{-1}$

- Complete Cholesky : K^{-1}
- Projected Incomplete Cholesky : $P_n\hat{K}^{-1}P_n$
- Projected Cholesky : $P_nK^{-1}P_n$

4 Application Examples

The Preconditioned Projected Arnoldi Algorithm is implemented in Python using the Scipy [15] sparse linear algebra library, which provides a wrapper for the Implicity Restarted Arnoldi (IRA) method implemented in ARPACK. The projection and preconditioner steps are carried-out by Scipy sparse linear operator. Three examples are considered to illustrate the algorithm proposed by this paper. In all cases, structural steel property is used ($E = 210GPa$, $\nu = 0.3$ and $= 7500Kg/m^3$). The PCG tolerance is set to 1.0^{-10} and maximum Arnoldi iterations n_{int} is set to 10 times the number of eigenpairs required. For all test-cases, the first 20 modes are computed for all possible nodal diameters.

The first example, named case A, consists of a 2D mesh, with plane stress triangle elements with 1m of thickness. The number of displacement dofs of the reference sector is 266, number of cyclic constraints is 14, and the whole geometry consists of 8 sectors. Table 1 shows for all nodal diameters the number of average iterations took by the PCG depending on the choice of the preconditioner. The average iterations is measured by the number of linear operator calls divided by the predefined number of modes to be computed by the IRA algorithm. Consequently, the average is measuring the overall performance of the eigensolver and not only the PCG performance. Figure 3 shows the first three modal families for nodal diameter 0,1 and 2.

Preconditioner	I	$\hat{K}^{-1}$	K^{-1}	$P_n\hat{K}^{-1}P_n$	$P_nK^{-1}P_n$
$n = 0$	165.64	42.77	18.95	42.26	18.42
$n = 1$	163.33	45.31	20	45.41	20.00
$n = 2$	162.48	46.30	20	46.13	19.97
$n = 3$	161.77	47.36	19.99	47.30	19.98

Table 1: Average PCG iterations per mode for case A.

Preconditioner	I	$\hat{K}^{-1}$	K^{-1}	$P_n\hat{K}^{-1}P_n$	$P_nK^{-1}P_n$
$n = 0$	491.08	95.35	29.92	94.89	29.95
$n = 1$	485.56	166.56	33.65	167.02	33.54
$n = 2$	481.02	175.24	33.78	174.62	33.76
$n = 3$	474.08	201.08	34.29	201.59	34.27

Table 2: Average PCG iterations per mode for case B.

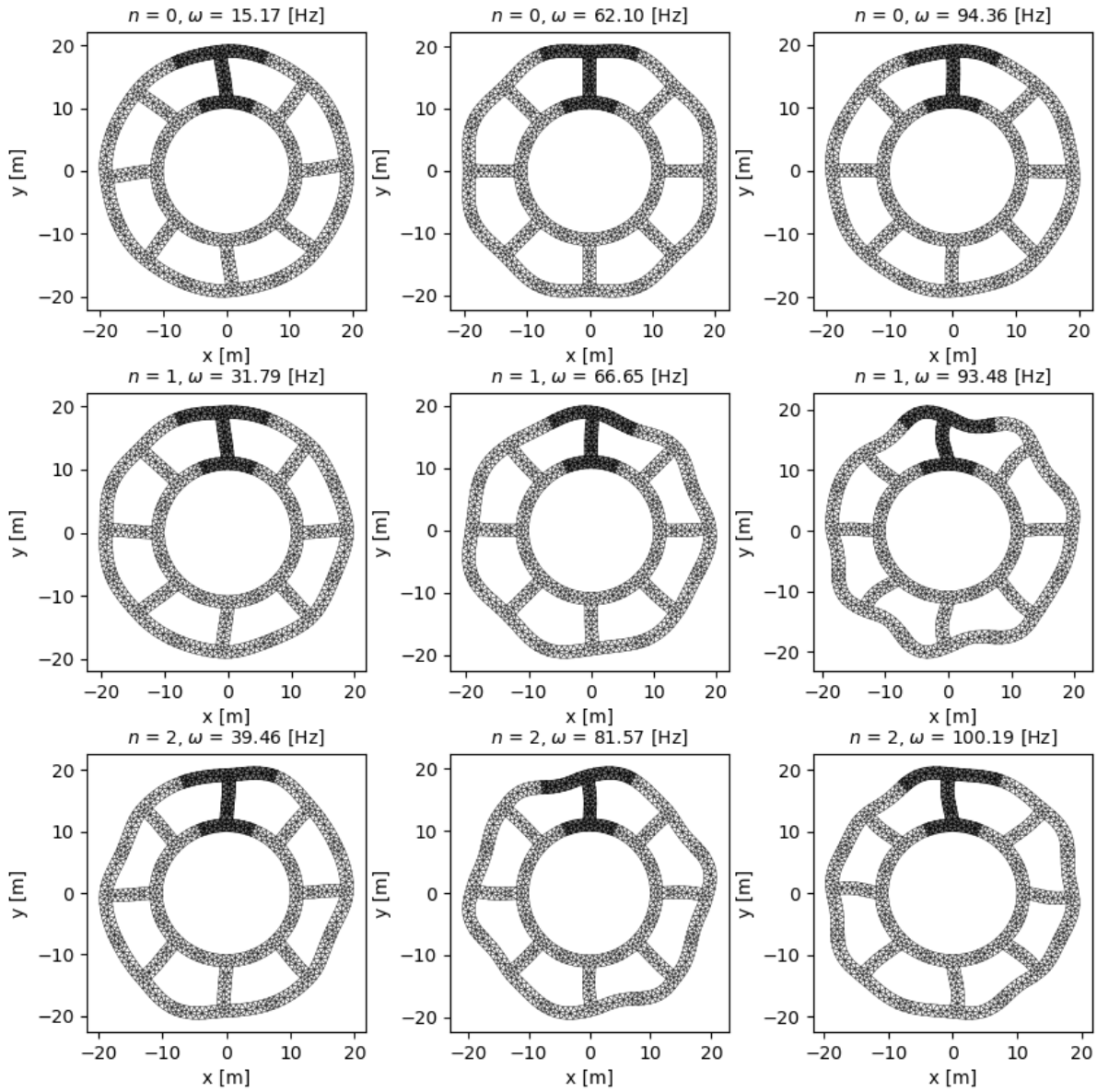

Figure 3: Mode shapes for case A.

The two remaining test cases, B and C, are both 3D Finite Element Models with 1596 and 1467 degrees of freedom respectively. Case B has a simple geometry composed by 8 sector, with 86 cyclic constraints, and the mesh contains only linear hexa elements. Figure 4 shows the first three modes for three nodal diameters and Table 2 summarizes the number of average iterations per mode using the five preconditioners. Case C is composed by tetra elements, 153 cyclic constraints, and it has a more realistic geometry with 24 sector, which resembles a blade-disk geometry. Figure 5 shows the first 3 modes for nodal diameter 0,1, and 2 for the case C. Table 3 presents the number of iteration by the PCG solver. It is clear that for all cases, the projection operation in the preconditioners, $P_n^H K^{-1} P_n$ and $P_n^H \hat{K}^{-1} P_n$ does not increase the performance of the PCG algorithm. Therefore, they must be avoid due to the additional computational cost. For case A and B, the precondition performance is almost constant with the

nodal diameters, whereas in the case C it is possible to see some discrepancies. For instance, the nodal diameter 1 and 4 converged with less than 81 iteration whereas $n = 2$ and $n = 3$ took more than 350 iterations to converge. This behavior can possibly be explained by the differences between free vibration modes without and with cyclic symmetry. If the complex cyclic symmetry constraint does not affect much the free vibration of the reference sector, then K^{-1} will be a good estimation of $(P_n^H K P_n)^{-1}$. Physically, this means that the projected and free modes are almost identical. This fact shows that finding efficient preconditioners for all nodal diameter indices remains a challenge. However, it might be possible to define a family of preconditioners which a more suitable for specific subsets of ND.

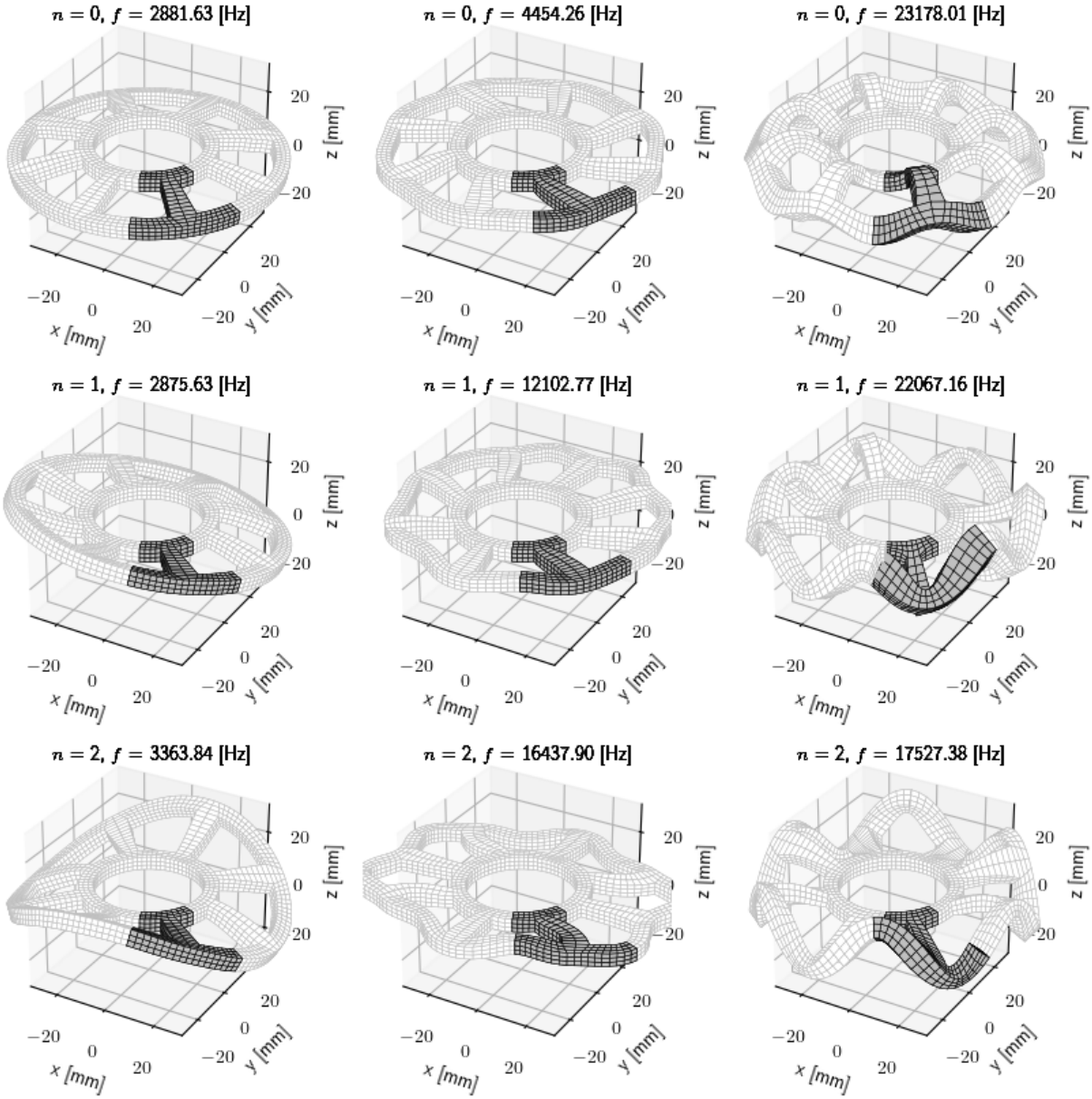

Figure 4: Mode shapes for case B.

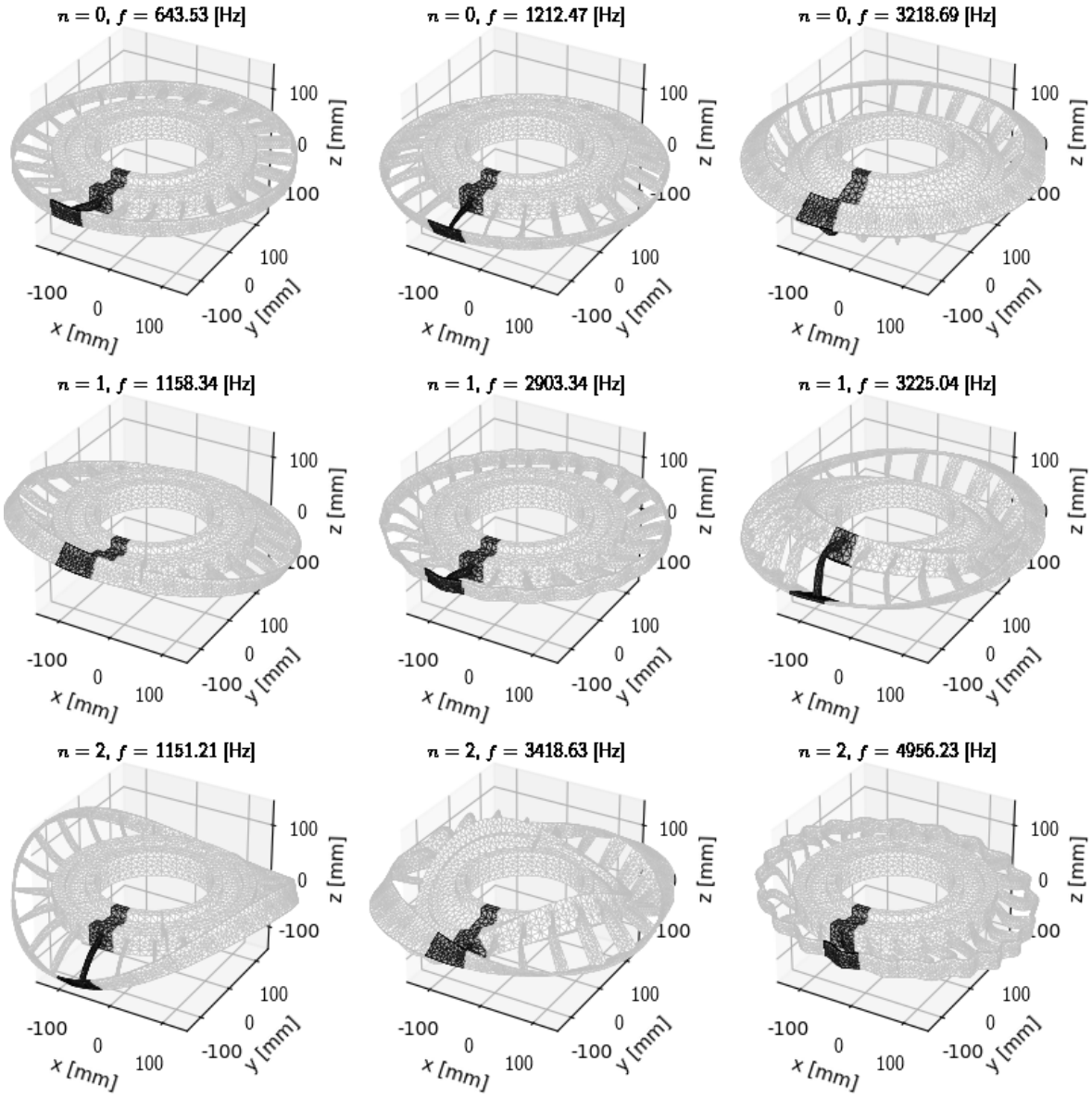

Figure 5: Mode shapes for case C.

Preconditioner	I	$\hat{K}^{-1}$	K^{-1}	$P_n\hat{K}^{-1}P_n$	$P_nK^{-1}P_n$
$n=0$	1676.8	568.3	526.4	568.0	526.4
$n=1$	1438.3	354.6	68.6	354.9	300.2
$n=2$	1421.8	409.1	352.2	360.1	303.5
$n=3$	1610.8	137.8	356.8	367.1	76.1
$n=4$	1371.3	142.7	80.3	142.0	310.4
$n=5$	1344.0	156.5	81.8	155.3	81.7
$n=6$	1323.4	162.4	86.4	163.4	86.7
$n=7$	1307.1	170.9	89.3	171.1	89.4
$n=8$	1295.0	178.3	91.0	178.6	90.7
$n=9$	1286.1	176.8	90.9	176.8	90.3
$n=10$	1272.0	182.5	92.8	181.7	92.8
$n=11$	1268.7	184.4	93.0	182.5	92.9
$n=12$	1266.9	179.9	92.6	180.7	92.7

Table 3: Average PCG iterations per mode for case C.

5 CONCLUSIONS

The cyclic symmetry constraints are presented using a substructuring framework, where primal and dual systems are formulated. These two approaches can be exploited and lead to different eigen-solvers strategies. The primal assembly formulation modifies the reference sector stiffness and mass matrices, which fully decouples the cyclic sectors by means of the unique set of primal variables. The $\bar{N}$ harmonic problems may be solved independently, but the similarity between the harmonic systems and the reference sector is usually not explored. On the other hand, the dual assembly formulation leads to a singular hybrid system of equations with the displacements and the Lagrange multipliers. In this case, the reference sector matrices, K and M are retained unchanged, whereas the complex cyclic constraint depends on the nodal diameter. Despite the singularity, the structure of the dual formulation allows to build a projected generalized eigenvalue problem. Using this technique, one can utilize the stiffness matrix of the reference sector, with some variations, as an efficient preconditioner. A total of five preconditioners namely, Identity (I), Incomplete Cholesky (IC), Complete Cholesky (CC), Projected Incomplete Cholesky (PIC) and Projected Cholesky (PC) were implemented and tested. The projected versions, PCI and PC, added unnecessary computational cost since they do not reduce the number of iterations when compared to IC and CC. The Cholesky preconditioner performed very well, where speed-ups could be seen in all developed test-cases when compared to pure CG (identity preconditioner). In the 2D case, the complete Cholesky is around 8 times faster than the identity preconditioner, whereas in the 3D case varies from 3 to 17 faster depending on the ND index. This suggests that finding suitable preconditioner for all nodal diameters spectrum is very challenging. Besides that, it is important to notice that, the incomplete Cholesky reveals an algorithmic trade-off with a less costly factorization, but more iterations in the PCG. In the future, it is intended to compare the eigen-solvers' computational efficiency for primal and dual formulation with respect to the number of eigen-pairs required. Furthermore, the projected Arnoldi algorithm will be extended to deal with blade-disk mistuning and quasi-cyclic geometries.

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